BOUNDS ON THE a-INVARIANT AND REDUCTION NUMBERS OF IDEALS

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Abstract. Let R be a d-dimensional standard graded ring over an Artinian local ring. Let $\mathfrak{M}$ be the unique maximal homogeneous ideal of R. Let $h^i(R)_n$ denote the length of the nth graded component of the local cohomology module $H^i_{\mathfrak{M}}(R)$. Define the Eisenbud-Goto invariant $EG(R)$ of R to be the number $\sum_{q=0}^{d-1}\binom{d-1}{q}h^q_{\mathfrak{M}}(R)_{1-q}$. We prove that the a-invariant $a(R)$ of the top local cohomology module $H^d_{\mathfrak{M}}(R)$ satisfies the inequality: $a(R) \leq e(R) - \ell(R_1) + (d-1)(\ell(R_0)-1) + EG(R)$. This bound is used to get upper bounds for the reduction number of an $\mathfrak{m}$-primary ideal I of a Cohen-Macaulay local ring $(R, \mathfrak{m})$, when the associated graded ring of I has depth at least $d-1$.

1. Introduction

Let $R = \bigoplus_{n=0}^{n=\infty} R_n$ be a d-dimensional standard graded ring over an Artinian local ring R_0. Let $\mathfrak{M}$ be the maximal homogeneous ideal of R. Let $H^i_{\mathfrak{M}}(R)$ denote the i-th local cohomology module of R with respect to $\mathfrak{M}$. For a graded module M, we use $[M]_n$ or M_n to denote the nth graded component of M. The a-invariant of R, introduced in [GW], is defined as

$$a(R) = \max\{n \,\big|\, [H^d_{\mathfrak{M}}(R)]_n \neq 0\}.$$

The objective of this paper is to give a bound for the a-invariant of R in terms of lengths of graded components of local cohomology modules and use it to get bounds for reduction numbers of ideals. Let $\ell(M)$ denote length of a module M. We set $\ell\left([H^q_{\mathfrak{M}}(R)]_{1-q}\right) = h^q(R)_{1-q}$ for all $q \geq 0$. To state our bound for the a-invariant we define the Eisenbud-Goto invariant $EG(R)$ of R to be the number

$$EG(R) = \sum_{q=0}^{d-1}\binom{d-1}{q} h^q(R)_{1-q}.$$

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The main result of the paper is:

Theorem 1.1. *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a d-dimensional standard graded algebra over an artinian local ring R_0 with multiplicity $e(R)$. Then*

$$a(R) \leq e(R) - \ell(R_1) + (d-1)(\ell(R_0) - 1) + EG(R).$$

Eisenbud and Goto [EG] showed that if R_0 is a field then

$$e(R) \geq 1 + \operatorname{codim}(R) - EG(R).$$

They showed that if equality holds in the above inequality then $R/H^0_{\mathfrak{M}}(R)$ has linear resolution. To state our bounds for reduction numbers we recall some basic concepts about reductions of ideals. Let $(R, \mathfrak{m})$ be a local ring. Let $J \subset I$ be ideals of R. The ideal J is called a reduction of I if there exists an $n \in \mathbb{N}$ such that $JI^n = I^{n+1}$ [NR]. Among the reductions of I, the smallest ones with respect to inclusion are called minimal reductions of I. If $R/\mathfrak{m}$ is infinite then any minimal reduction of I is minimally generated by as many elements as the Krull dimension of the fiber cone $F(I) := \bigoplus_{n=0}^{n=\infty} I^n/\mathfrak{m}I^n$. The *reduction number*, $r_J(I)$, of I with respect to a minimal reduction J is the least integer n for which $JI^n = I^{n+1}$. When $R/\mathfrak{m}$ is infinite, the reduction number of I is defined as the minimum of the reduction numbers $r_J(I)$ where J varies over all the minimal reductions of I. Let $G(I) := \oplus_{n \geq 0} I^n/I^{n+1}$ be the associated graded ring of an ideal I. Let $\gamma(I)$ denote the depth of the irrelevant ideal G_+ of $G(I)$. If $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, I is an $\mathfrak{m}$-primary ideal and $\gamma(I) \geq d-1$, then $r(I) = a(G(I)) + d$ [M]. The *Ratliff-Rush closure* of an ideal I, $\widetilde{I}$, is the stable value of the sequence of the ideals $\{I^{n+1} : I^n\}$. We will obtain the following bounds for $r(I)$ as an application of the main theorem:

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal with $\gamma(I) \geq d-1$. Let J be any minimal reduction of I.*

1. *Let $d = 1$. Then $r(I) \leq e(I) - \Big(\ell(I/(I \cap \widetilde{I^2}) - 1\Big) \leq e(I)$.*
2. *Let $d = 2$. Put $X = Proj(G(I))$. Then $r(I) \leq 1 + e(I) - \ell(I/I^2) + \ell(H^0(X, \mathcal{O}_X))$.*
3. *Let $d \geq 3$. Then $r(I) \leq 1 + \ell(I^2/JI) + h^{d-1}(G(I))_{2-d}$.*

We will show by an example that our bounds for the a-invariant and reduction number are sharp.

2. A bound on the a-invariant of standard graded algebras

In this section we prove our bound on the a-invariant of a standard graded algebra R over an Artinian local ring R_0.

Theorem 2.1. *Let $R = \bigoplus_{n=0}^{n=\infty} R_n$ be a d-dimensional standard graded algebra over an Artinian local ring R_0. Then*

$$a(R) \leq e(R) - \ell(R_1) + (d-1)(\ell(R_0) - 1) + EG(R). \tag{1}$$

Proof. We may assume without loss of generality that the residue field of R_0 is infinite. We prove the theorem by induction on d. Let $d = 0$. Then

$$e(R) = \ell(R_0) + \ell(R_1) + \cdots + \ell(R_m),$$

where $m = a_0(R)$. Thus

$$e(R) - \ell(R_1) - \ell(R_0) + 1 = 1 + \ell(R_2) + \cdots + \ell(R_m) \geq m.$$

Let R be Cohen-Macaulay and pick a degree one nonzerodivisor x to see that

$$\begin{aligned} a(R) &= a(R/xR) - 1 \\ &\leq e(R/xR) - \ell\left([R/xR]_1\right) + (d-2)(\ell(R_0) - 1) - 1 \\ &= e(R) - \ell(R_1) + \ell(R_0) + (d-2)(\ell(R_0) - 1) - 1 \\ &= e(R) - \ell(R_1) + (d-1)(\ell(R_0) - 1). \end{aligned}$$

Now let $d = 1$. If R is Cohen-Macaulay , we are done by the above argument. So let $\operatorname{depth}(R) = 0$. Then $S := R/H^0_{\mathfrak{M}}(R)$ is Cohen-Macaulay , $e(S) = e(R)$ and $a(R) = a(S)$. Hence

$$a(R) = a(S) \leq e(S) - \ell(S_1) = e(R) - \ell(R_1) + h^0(R)_1.$$

Suppose $d \geq 2$. Let $x \in R_1$ be a superficial element. We first prove that for a degree one superficial element in R,

$$EG(R/xR) \leq EG(R).$$

Since x is superficial of degree one,

$$H^i_{\mathfrak{M}}(0 :_R x) = \begin{cases} (0 :_R x) & \text{if } i = 0 \\ 0 & \text{if otherwise.} \end{cases} .$$

Hence from the short exact sequence

$$0 \longrightarrow (0:_R x) \longrightarrow R \longrightarrow \frac{R}{(0:_R x)} \longrightarrow 0$$

we get $H^i_{\mathfrak{M}}(R/(0:_R x)) = H^i_{\mathfrak{M}}(R)$ for all $i \geq 1$. From the exact sequence

$$0 \longrightarrow \frac{R}{(0:_R x)}(-1) \longrightarrow R \longrightarrow \frac{R}{xR} \longrightarrow 0$$

we get the long exact sequence

$$\begin{aligned} \cdots \longrightarrow [H^i_{\mathfrak{M}}(R)]_n \longrightarrow [H^i_{\mathfrak{M}}(R/xR)]_n \longrightarrow [H^{i+1}_{\mathfrak{M}}(R)]_{n-1} \\ \longrightarrow [H^{i+1}_{\mathfrak{M}}(R)]_n \longrightarrow \cdots . \end{aligned}$$

Hence for all $i \geq 0$,

$$h^i(R/xR)_n \leq h^i(R)_n + h^{i+1}(R)_{n-1}.$$

Hence

$$\begin{aligned} EG(R/xR) &= \sum_{q=0}^{d-2} \binom{d-2}{q} h^q(R/xR)_{1-q} \\ &\leq \sum_{q=0}^{d-2} \binom{d-2}{q} \left[h^q(R)_{1-q} + h^{q+1}(R)_{-q}\right] \\ &= \sum_{q=0}^{d-2} \binom{d-2}{q} h^q(R)_{1-q} + \sum_{q=1}^{d-1} \binom{d-2}{q-1} h^q(R)_{1-q} \\ &= \sum_{q=0}^{d-1} \binom{d-1}{q} h^q(R)_{1-q} \\ &= EG(R). \end{aligned}$$

Therefore

$$\begin{aligned} a(R) &\leq a(R/xR) - 1 \quad \text{by [T]} \\ &\leq e(R/xR) - \ell(R/xR)_1 + (d-2)\left(\ell\left([R/xR]_0\right) - 1\right) + EG(R) - 1 \\ &= e(R) - \ell(R_1) + \ell(R_0) - \ell\left((0:x)_{R_0}\right) + (d-2)\left(\ell(R_0) - 1\right) \\ &+ EG(R) - 1 \\ &\leq e(R) - \ell(R_1) + (d-1)\left(\ell(R_0) - 1\right) + EG(R). \end{aligned}$$

□

We now demonstrate that the bound in Theorem 2.1 is sharp.

Example 2.2. Let k be a field and x, y, a, b, c, d be indeterminates. Consider the ideal $I = (x^3, x^2y^4, xy^5, y^7)$ in the polynomial ring $S = k[x, y]$. Using Hilbert series we show that $F(I) \simeq k[a, b, c, d]/(bd, bc, b^2, c^3)$. Consider the ring homomorphism $\phi : R = k[a, b, c, d] \longrightarrow F(I)$ defined by

$$\phi(a) = \overline{x^3}, \ \ \phi(b) = \overline{x^2y^4}, \ \ \phi(c) = \overline{xy^5}, \ \text{ and } \ \phi(d) = \overline{y^7}.$$

Here the overbar indicates the image in the first graded component of $F(I)$. Let $L = \ker\, \phi$. The equations

$$\begin{aligned} (x^2y^4)(y^7) &= (xy^5)^2y, \\ (x^2y^4)(xy^5) &= (x^3y^7)y^2, \\ (x^2y^4)^2 &= (x^3)(xy^5)y^3, \\ (xy^5)^3 &= x^3(y^7)^2y, \end{aligned}$$

show that $N = (bd, bc, b^2, c^3) \subset L$. To show that $N = L$, we show that R/N and R/L have same Hilbert series. We denote the Hilbert series of a graded algebra G by $H(G, \lambda)$. By the propositions 2.3 and 2.6 of [H] we find that $\mu(I^n) = 3n + 1$ for all $n \geq 0$. Here μ denotes the minimum number of generators. This shows that $H(F(I), \lambda) = (1 + 2\lambda)/(1 - \lambda)^2$. By the well known "divide and conquer strategy" for finding Hilbert series of quotients of polynomial rings by monomial ideals we get, $H(R/N, \lambda) = H(F(I), \lambda)$. Thus $F(I) \cong R/N$. Therefore $F(I)$ is a two - dimensional ring with depth one. Notice that $N = (b, c^3) \cap (c, d, b^2)$. Put $J = (b, c^3)$ and $K = (c, d, b^2)$. In order to get the desired information about local cohomology of $F(I)$, consider the exact sequence :

$$0 \longrightarrow F(I) \longrightarrow R/J \bigoplus R/K \longrightarrow R/(J + K) \longrightarrow 0.$$

Hence we get the following long exact sequence of local cohomology modules with respect to the maximal homogeneous ideal $\mathfrak{m} = (a, b, c, d)$:

$$0 \longrightarrow H^1_{\mathfrak{m}}(F(I)) \longrightarrow H^1_{\mathfrak{m}}(R/K)$$

$$\longrightarrow H^1_{\mathfrak{m}}(R/(J + K)) \longrightarrow H^2_{\mathfrak{m}}(F(I)) \longrightarrow H^2_{\mathfrak{m}}(R/J) \longrightarrow 0.$$

We now show that $a(F(I)) = 0$ and $h^1(F(I))_0 = 1$. Since

$$R/J \simeq k[a, c, d]/(c^3), \ \ R/(J + K) \simeq k[a] \ \text{ and } \ R/K \simeq k[a, b]/(b^2),$$

by using that fact that $a(R/(f)) = a(R) + \deg(f)$ for a homogeneous regular element f of a graded algebra R, we conclude that $a(R/J) = 0, \quad a(R/(J + K)) =$

-1 and $a(R/K)=0$. Thus $a(F(I))=0$. By [BH, Theorem 4.4.3], we get

$$h^1(F(I))_0 = h^1(R/K)_0 = P(R/K,0) - H(R/K,0) = 2-1 = 1.$$

Substituting these values in (1) we observe that equality holds. Therefore (1) is sharp. $\square$

3. Bounds on reduction numbers

In this section we will use the bound on the a-invariant obtained in the previous section to provide bounds on reduction numbers. By [T] and [M], we know that $r(I) = a(G(I)) + d$ where $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d and $\gamma(I) \geq d-1$. We will consider the cases where $d=1$, $d=2$ and $d \geq 3$ separately. In the next result we will need the formula: $[H^0_{G_+}(G(I))]_n = (I^n \cap \widetilde{I^{n+1}})/I^{n+1}$ for all $n \geq 0$ [HJLS].

Proposition 3.1. *Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring. Let I be an $\mathfrak{m}$-primary ideal. Then*

$$r(I) \leq e(I) - \left[\ell(I/(I \cap \widetilde{I^2}) - 1\right] \leq e(I).$$

Proof. Since $d=1$,

$$\begin{aligned} a(G(I)) &\leq e(I) - \ell(I/I^2) + h^0(G)_1 \\ &= e(I) - \ell(I/I^2) + \ell((I \cap \widetilde{I^2})/I^2) \\ &= e(I) - \ell(I/(I \cap \widetilde{I^2})). \end{aligned}$$

Hence $r(I) = a(G(I)) + 1 \leq 1 + e(I) - \ell(I/(I \cap \widetilde{I^2})$. If $\ell(I/(I \cap \widetilde{I^2}) = 0$ then $I \subseteq \widetilde{I^2}$. But $\widetilde{I^2} = I^{n+2} : I^n$ for large n. Hence $I^{n+1} = I^{n+2}$. This is a contradiction. Hence $\ell(I/(I \cap \widetilde{I^2})) \geq 1$. Thus we obtain the classical bound $r(I) \leq e(I)$. $\square$

Example 3.2. Let k be a field and t be an indeterminate. Put $R = k[[t^4, t^5, t^6, t^7]]$ and $I = (t^4, t^5, t^6)$. Let $\mathfrak{m}$ denote the unique maximal ideal of R. Let G denote the associated graded ring $G(I)$ of I. Then G is not Cohen-Macaulay since $t^7 I \subset I^2$. To find the associated Ratliff-Rush ideal of I notice that $I^2 = \mathfrak{m}^2$. Since $r(\mathfrak{m}) = 1$, the associated graded ring $G(\mathfrak{m})$ is Cohen-Macaulay by [S]. Therefore all powers of $\mathfrak{m}$ are Ratliff-Rush. Hence, $\widetilde{I^2} = \widetilde{\mathfrak{m}^2} = \mathfrak{m}^2 = I^2$. Hence $(I \cap \widetilde{I^2}/I^2) = 0$. Therefore $r(I) \leq 1 + e(I) - \ell(I/I^2) = 2$. It can be checked that $r_{(t^4)}(I) = 2$. Therefore the bound in the above result is sharp.

Proposition 3.3. *Let I be an $\mathfrak{m}$-primary ideal of a two dimensional Cohen-Macaulay local ring with $\gamma(I) \geq 1$. Let $X = Proj\ G(I)$. Then*

$$r(I) \leq 1 + e(I) - \ell(I/I^2) + \ell(H^0(X, \mathcal{O}_X)).$$

Proof. Since $\gamma(I) \geq 1$,

$$r(I) \leq 1 + e(I) - \ell(I/I^2) + \ell(R/I) + h^1(G)_0.$$

By the exact sequence

$$0 \longrightarrow H^0_{G_+}(G) \longrightarrow G \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n)) \longrightarrow H^1_{G_+}(G) \longrightarrow 0$$

we get, by taking the $0th$ component of all the modules in the above exact sequence:

$$\ell(H^0(X, \mathcal{O}_X)) - \ell(R/I) = h^1(G)_0$$

Putting this in the above bound for $r(I)$ we get the desired upper bound. □

Proposition 3.4. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d \geq 3$ with $R/\mathfrak{m}$ infinite. Let I be an $\mathfrak{m}$-primary ideal with a minimal reduction J and $\gamma(I) \geq d-1$. Then*

$$r(I) \leq 1 + \ell(I^2/JI) + h^{d-1}(G)_{2-d}.$$

Proof. Since $\gamma(I) \geq d-1$, by the Theorem 2.1

$$\begin{aligned} r(I) &\leq e(I) - \ell(I/I^2) + (d-1)(\ell(R/I) - 1) + h^{d-1}(G)_{2-d} + d \\ &= \ell(R/J) - \ell(R/I^2) + \ell(R/I) + (d-1)\ell(R/I) \\ &- (d-1) + d + h^{d-1}(G)_{2-d} \\ &= \ell(R/J) - \ell(R/I^2) + \ell(J/JI) + 1 + h^{d-1}(G)_{2-d} \\ &= 1 + \ell(I^2/JI) + h^{d-1}(G)_{2-d} \end{aligned}$$

□

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